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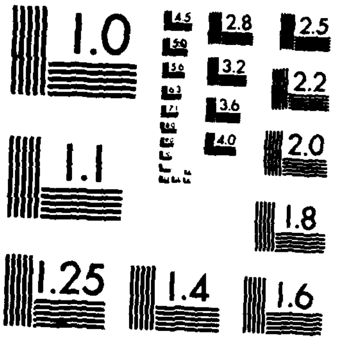

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On Vorticity Boundary Conditions

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ABSTRACT

We discuss the problem of boundary conditions for the vorticity form of the 2-D Navier-Stokes equations and the Prandtl boundary layer equations. We present a new formulation of the vorticity boundary conditions and relate these conditions to those which are currently used in numerical algorithms. We present a finite difference scheme incorporating the new boundary conditions for the Prandtl boundary layer equations. Numerical results are presented.

1. Vorticity Boundary Conditions for the 2-D Navier-Stokes Equations

In this paper we present some preliminary results concerning investigations into the issue of boundary conditions for vorticity and the boundary generation of vorticity in the context of the two dimensional incompressible Navier-Stokes equations and the Prandtl boundary layer equations. We first discuss the origin of the problem with vorticity boundary values for the Navier-Stokes equations. We then discuss two techniques, one due to Chorin [3], and the other due to Quartapelle and Valz-Gris [8], for overcoming this problem. These two techniques seem very *different*, and yet when incorporated into numerical schemes, both appear to give reasonable solutions to the Navier Stokes equations. One of the primary motives for this investigation was to understand why this should be so. In order to obtain this understanding, we found it necessary to derive our own solution to the problem of boundary conditions. We present this, and use the derivation to help explain the relationship between the two earlier approaches. Our derivation also suggests new methods for implementing the vorticity boundary conditions in numerical schemes. This aspect will be discussed in a future paper, however, to illustrate the general idea of our approach we consider an application of our technique to the Prandtl boundary layer equations. We present a numerical scheme for these equations and provide some computational results obtained with this scheme. We also compare a finite difference scheme for these equations with a scheme introduced by Chorin in [4].

The equations we are concerned with are the two dimensional incompressible Navier-Stokes equations,

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\nabla P + \nu \Delta u \tag{1.1}$$

$$\nabla \cdot \vec{u} = 0 \tag{1.2}$$

$$\vec{u}(\alpha) = \vec{B}(\alpha) \quad \text{for } \alpha \in \partial\Omega \ . \tag{1.3}$$

Here $\vec{u}$ is the velocity, P the pressure and ν the viscosity. We assume the fluid is of constant density equal to one. Ω is the region in R^2 with boundary $\partial\Omega$. $\vec{B}(\alpha)$ is the velocity on the boundary, often taken to be identically zero.

In the vorticity formulation of (1.1)-(1.3) the velocity field $\vec{u}$ is taken to be the sum of a velocity field due to an irrotational flow and a velocity field due to a rotational flow. Let ϕ be the potential for the irrotational flow and Ψ the stream function for the rotational flow. We assume ϕ satisfies

$$\Delta\phi = 0 \qquad \frac{\partial\phi}{\partial\vec{n}} = \vec{B}\cdot\vec{n} \quad \text{on } \partial\Omega \ .$$

Here $\vec{n}$ denotes the normal to the boundary

Let ω be the vorticity. By taking the curl of equation (1.1) one obtains the following equation for the transport of vorticity,

$$\frac{\partial\omega}{\partial t} + \mathrm{U}\cdot\nabla\omega = \nu\Delta\omega \ . \tag{1.4}$$

Here

$$\mathrm{U} = (\phi_x, \phi_y) + (\Psi_y, -\Psi_x) \tag{1.5}$$

and Ψ is determined from the equation

$$\begin{gathered} \Delta\Psi = -\omega \\ \Psi = 0 \quad \text{on } \partial\Omega \ . \end{gathered} \tag{1.6}$$

The boundary conditions given above for ϕ and Ψ guarantee that the normal velocity boundary condition is satisfied. To satisfy the tangential velocity condition we must also have

$$\frac{\partial\Psi}{\partial n} = \vec{B}\cdot\vec{t} - (\phi_x,\phi_y)\cdot\vec{t} = b(x) \quad \text{for } x \in \partial\Omega \tag{1.7}$$

where $\vec{t}$ is the unit tangent to $\partial\Omega$.

Often the vorticity form of the equations is used in numerical computations. One reason for its use is that the number of unknowns is reduced from three to one. Also, in many flows the vorticity is itself of great interest and it is desirable to compute it's evolution directly rather than compute in the primitive variables and differentiate the result. The use of the vorticity formulation is not without it's difficulties. One major difficulty is the proper vorticity boundary conditions.

In the translation of the equations from primitive variables (1.1)-(1.3) to the vorticity form (1.4)-(1.7), boundary conditions for the vorticity are not obtained. Moreover, it appears that too many boundary conditions are given on the stream function. (Both those in (1.6) and (1.7) must be satisfied.) However, a bit of thought reveals that the freedom in choosing the vorticity boundary conditions occurs precisely because the stream function is over determined. For, unless there is some mechanism for manipulating ω in the interior of the domain, the problem which defines Ψ will not, in general, be well posed. It is therefore not surprising that a common thread which runs through all numerical implementations is the manipulation of the vorticity, usually near the boundary, in a way which insures that both (1.6) and (1.7) are satisfied. However, apart from this similarity, the numerical techniques for overcoming this difficulty can appear quite different. We shall consider two approaches in use. There are others, most notably those which use the relation between Ψ and ω in (1.6) and (1.7) at points on the boundary. For a review of these, and of the results which can be obtained with them, see [6] and [7].

One approach to solving the problem of boundary conditions is that which is implicitly used by Chorin in [3]. In that paper, he utilizes the method of fractional steps to solve equations (1.4)-(1.7). The method is to advance for one timestep the solution to the Euler equations:

$$\frac{\partial \omega}{\partial t} + \tilde{U} \cdot \nabla \omega = 0 \qquad (1.8)$$

(with U defined by equation (1.5) and Ψ determined only by (1.6)) followed by one timestep of the solution of the heat equation

$$\frac{\partial \omega}{\partial t} = \nu \Delta \omega \quad . \tag{1.9}$$

The vortex blob method is used to advance the solution for the Euler step, and a random walk technique is used to simulate the diffusion step. In the solution of the Euler equations, (1.8), boundary conditions on the vorticity are not needed since $\mathrm{U} \cdot \hat{n} = 0$. In the solution of the heat equation (1.9) boundary conditions are needed, and one of the novel ideas contained in Chorin's paper is the method by which the boundary values of vorticity are obtained.

After one step of the Euler equations, the interior vorticity distribution will not necessarily induce a velocity field which satisfies the tangential velocity boundary condition. If $\tilde{\omega}$ is the distribution of vorticity after one time step of Euler, and $(\tilde{u}, \tilde{v})$ is the velocity constructed from this distribution, then $(\tilde{u}, \tilde{v}) \cdot \hat{\tau} \neq b(s)$ for $s \in \partial\Omega$. Chorin's idea is to introduce vorticity on the boundary in an amount which cancels this error. Specifically, if the boundary is divided up into segments of length h then the vorticity introduced per segment is

$$\omega(s_i) = (\tilde{u}, \tilde{v})|_{s_i} \cdot \hat{\tau}\, h - b(s_i)$$

where s_i is the center of the ith segment on the boundary. This vorticity is then allowed to diffuse into the region and participate in the evolution of the flow.

A common concept in fluid mechanics is that vorticity is "produced" at object boundaries. As discussed by Batchelor in [2], this production occurs so that the external or outer fluid velocity will match the velocity of the object. An attractive feature of Chorin's method is that it mimics this process of boundary generation of vorticity. Computations performed with this method (and an improved version incorporating boundary layer modelling [4]) suggest that this algorithm generates a good approximation to the solution of the equations (1.4)-(1.7). From this fact we conclude that boundary vorticity generation, as understood from the nature of the physical problem, is intimately related to the missing boundary conditions for the vorticity transport equations.

The second approach is that due to Quartapelle and Valz-Gris [8] and is more mathematically rather than physically motivated. Essentially, they pose the following question "What are the conditions on the vorticity so that the stream function constructed using (1.6) also satisfies (1.7)?" The answer is contained in the following theorem presented in their paper,

Theorem. $\Delta\Psi = -\omega$ in Ω, $\Psi|_s = a(s)$ and $\frac{\partial\Psi}{\partial\vec{n}} = b(s)$, if and only if

$$\int_{\Omega} \omega\ \eta da = \int_{\partial\Omega} \eta b(s) - a(s)\frac{\partial\eta}{\partial n} ds \tag{1.10}$$

for any function η harmonic in Ω.

In particular, if $a(s) = 0$ and a no slip condition is specified, $b(s) = 0$, then the vorticity must be orthogonal to all harmonic functions defined in the domain.

Using this result, Quartapelle and Valz-Gris overcome the difficulty of finding explicit boundary conditions on the vorticity by adjoining the constraint (1.10) to the equations for vorticity transport (1.4). Their implementation consists of advancing the solution of (1.4) one time step without regard to the constraint and then projecting the result onto the component which satisfies (1.10). It is for this reason that we refer to their approach as the projection method. Details of their numerical implementation and some nice computational results concerning the driven cavity problem are presented in [9].

Aside from the fact that both approaches are used to solve the same set of equations, the techniques do not appear to be related. Chorin's scheme effects the vorticity on the boundary, while in the method of Quartapelle and Valz-Gris, the interior vorticity is effected as well. In order to relate the two, we found it necessary to obtain a precise set of boundary conditions for the vorticity. Our derivation is motivated by the observation of Quartapelle and Valz-Gris that one should consider the evolution of the vorticity as a constrained evolution. Instead of explicitly finding the constraint on the vorticity and adjoining it to the equations as Quartapelle and Valz-Gris do, we precede by finding conditions which insure that as the vorticity evolves the constraint will automatically be satisfied.

We begin by expressing the "extra" boundary condition (1.7) as a constraint on the vorticity. For this purpose we use the Green's function, $G(x,s)$, for the domain Ω. This function is the solution of

$$\Delta G(x,s) = \delta(s) \quad x \in \Omega \quad G(x,s) = 0 \quad x \in \partial\Omega$$

where $\delta(s)$ is a Dirac delta function located at $s \in \Omega$. We have

$$\Psi(x) = -\int_{\Omega} G(x,s)\omega(s,t)ds$$

so that condition (1.7) can be written as

$$\frac{\partial}{\partial n}\int_{\Omega} G(x,s)\omega(s,t)ds = -b(x) \quad \text{for } x \in \partial\Omega \tag{1.11}$$

To find boundary conditions which insure that solutions of (1.4) induce a stream function which satisfies both (1.6) and (1.7), we find boundary conditions which guarantee that the derivative with respect to time of the constraint (1.11) vanishes. We require

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial n}\int_{\Omega} G(x,s)\omega(s,t)ds + b(x)\right) = 0 \quad \text{for } x \in \partial\Omega$$

If we use the fact that ω is a solution of (1.4) and the Green's identity

$$\omega = \int_{\Omega} G(x,s)\Delta\omega(s,t)ds - \int_{\partial\Omega}\frac{\partial G}{\partial n}(x,\alpha)\omega(\alpha,t)d\alpha$$

we find

$$\frac{\partial}{\partial t}\frac{\partial}{\partial n}\int_{\Omega} G(x,s)\omega(s,t)ds = 0$$

$$\rightarrow \frac{\partial}{\partial n}\int_{\Omega} G(x,s)\frac{\partial\omega}{\partial t}(s,t)ds = 0$$

$$\rightarrow \frac{\partial}{\partial n}\int_{\Omega} G(x,s)(-\vec{U}\cdot\nabla\omega(s,t) + \nu\Delta\omega(s,t))ds = 0$$

$$\rightarrow \nu\frac{\partial}{\partial n}\int_{\Omega} G(x,s)\Delta\omega(s,t)ds = \frac{\partial}{\partial n}\int_{\Omega} G(x,s)U \cdot \nabla\omega(s,t)ds$$

$$\rightarrow \frac{\partial\omega}{\partial n} + \frac{\partial}{\partial n}\int_{\partial\Omega}\frac{\partial G}{\partial n}(x,\alpha)\omega(\alpha,t)d\alpha = \frac{1}{\nu}\frac{\partial}{\partial n}\int_{\Omega} G(x,s)U \cdot \nabla\omega(s,t)ds \tag{1.12}$$

This is an integral-differential equation which determines the boundary values. Values satisfying (1.12) and used as data for (1.4) will insure that the time derivative of (1.11) will be zero. If we assume that the initial vorticity satisfies (1.11), then we will have insured that this constraint, and hence (1.7), is satisfied for all time.

If one uses the boundary conditions (1.12) with (1.4)-(1.6) then the resultant vorticity will also satisfy (1.7), and hence will satisfy the constraint as formulated by Quartapelle and Valz-Gris, (1.10). Conversely if one solves (1.4) with (1.10) explicitly satisfied then the vorticity will satisfy (1.12). We conclude that Quartapelle and Valz-Gris, by enforcing (1.10) explicitly at every timestep, are implicitly implementing the boundary conditions (1.12).

The connection between the boundary conditions (1.12) with the technique due to Chorin is less clear. If one considers finite difference methods for (1.4)-(1.6) using (1.12), then one finds that they have the structure that the boundary vorticity is obtained by solving an integral equation with a forcing function which is proportional to the error in the tangential velocity induced by one step of Euler flow. Rather than get into the details of the numerical schemes, this result can by illustrated by considering an explicit scheme which is discrete in time and continuous in space:

$$\frac{\omega^{k+1} - \omega^{k}}{\delta t} = -U \cdot \nabla\omega^{k} + \nu\Delta\omega^{k} \tag{1.13}$$

Boundary values of vorticity are needed in the evaluation of the Laplacian on the right hand side of (1.13). These values can be approximated using (1.12). The right hand side of (1.12) is evaluated using the solution at time $k\delta t$ and then the left hand side is solved for the boundary values. If one uses the expression for one step of Euler flow

$$\frac{\tilde{\omega}^{k+1}-\omega^{k}}{\delta t} = -\mathrm{U}\cdot\nabla\omega^{k} \tag{1.14}$$

to approximate the convective terms in the right hand side of (1.12), one finds

$$\frac{1}{\nu}\frac{\partial}{\partial n}\int_{\Omega} \mathrm{G}(x,s)\mathrm{U}\cdot\nabla\omega^{k}(s,t)ds = \frac{-1}{\nu}\frac{\partial}{\partial n}\int_{\Omega} \mathrm{G}(x,s)(\frac{\tilde{\omega}^{k+1}-\omega^{k}}{\delta t})ds$$

$$= \frac{-1}{\nu\delta t}(\frac{\partial}{\partial n}\int_{\Omega} \mathrm{G}(x,s)\;\tilde{\omega}^{k+1}ds - \frac{\partial}{\partial n}\int_{\Omega} \mathrm{G}(x,s)\;\omega^{k})ds$$

Thus the right hand side of (1.12) has the interpretation of being the difference in the tangential velocity at the boundary between that at time $k\delta t$ and that induced by one step of Euler flow. If the tangential velocity boundary condition is satisfied at time $k\delta t$, then the right hand side is just proportional to the error in velocity induced by one step of Euler flow.

The incorporation of boundary conditions (1.12) has the flavor of that utilized by Chorin, however, in Chorin's scheme one does not solve an integral equation to determine the boundary vorticity. We imagine that Chorin's method of constructing the strength is an approximation to the inversion of the integral equation on the left hand side of (1.12). A detailed analysis has yet to be carried out. For a related set of equations, the Prandtl boundary layer equations, it is very clear that Chorin's vorticity creation is equivalent to the implementation of vorticity boundary conditions. For a further discussion of this latter case, see the following section.

To conclude, one should view the process of solving the Navier-Stokes equations as a problem of solving a set of equations subject to a constraint. Quartapelle and Valz-Gris's technique is to explicitly adjoin the constraint to the vorticity evolution equations. Their numerical implementation takes the form of a projection method. Alternatively, one can incorporate the boundary conditions derived above. These conditions will insure that if the constraint is satisfied initially, then it will be satisfied for all time. We believe Chorin's scheme to be an approximation to these boundary conditions. Thus the schemes have the same goal, that of evolving the vorticity in such a way that the stream function satisfy boundary conditions (1.6) and (1.7), but they differ in the manner in which this goal is achieved. This observation explains why such different techniques applied to the same set of equations can both yield reasonable solutions.

2. Results for the Prandtl Boundary Layer Equations

In this section we apply the ideas of the first section to the Prandtl boundary layer equations. This set of equations, which we shall use to describe laminar flow over a half-infinite flat plate, is a simpler set of equations than the 2-D Navier Stokes equations, and yet when expressed in the vorticity formulation possesses the same problems with vorticity boundary conditions. We shall derive the continuous boundary conditions for the vorticity, and then implement a numerical method using these boundary conditions. Results of computations with this numerical method will be presented. A similar discussion concerning the implementation of numerical methods for the 2-D Navier Stokes equations will be presented in a forth coming paper [1].

In the vorticity form, the Prandtl boundary equations are

$$\frac{\partial \omega}{\partial t} + \vec{u} \cdot \nabla \omega = \nu \frac{\partial^2 \omega}{\partial y^2} \tag{2.1}$$

$$\omega = -\frac{\partial u}{\partial y} \tag{2.2}$$

$$u = U_0 + \int_y^\infty \omega(x,s,t)\,ds \tag{2.3}$$

$$v = -\int_0^y \frac{\partial u(x,s,t)}{\partial x}\,ds \tag{2.4}$$

with boundary conditions

$$u = v = 0 \quad \text{at } y = 0 \quad x > 0 \tag{2.5}$$

$$u = U_0 \quad \text{at } y = +\infty \tag{2.6}$$

Here (u,v) is the velocity and ω the vorticity. The domain is the quarter plane $0 \leq x \leq \infty$ and $0 \leq y \leq \infty$.

As is the case with the Navier Stokes equations, when one reconstructs the velocity field from the vorticity, it is not automatic that the boundary condition on u in (2.5) will be satisfied. Similarly, boundary conditions necessary to close equation (2.1) are not given. We precede, as in the first section, to derive boundary conditions by expressing the condition on u in (2.5) as a constraint on the vorticity and then setting time derivative of this constraint equal to zero. Using (2.3), the constraint on the vorticity is

$$U_0 + \int_0^\infty \omega(x,s,t)ds = 0 \quad x > 0 \quad \text{and } t \geq 0 \qquad (2.7)$$

Thus we require

$$\frac{\partial}{\partial t}\int_0^\infty \omega(x,s,t)ds = 0$$

$$\rightarrow \int_0^\infty \frac{\partial \omega}{\partial t}(x,s,t)ds = 0$$

$$\rightarrow \int_0^\infty -\vec{u} \cdot \nabla\omega + \nu\frac{\partial^2\omega}{\partial y^2}ds = 0$$

$$\rightarrow -\frac{\partial \omega}{\partial y}\Big|_{y=0} = \frac{1}{\nu}\int_0^\infty \vec{u} \cdot \nabla\omega ds \quad . \qquad (2.8)$$

(2.8) is the desired boundary condition. We now consider a numerical method for solving these equations which incorporates these boundary conditions.

Our computational domain is the rectangular region described by the points (x, y) such that $0 \leq x \leq x_\infty$ and $0 \leq y \leq y_\infty$. The mesh we use is rectangular with widths dx and dy in the x and y direction respectively. The values of the vorticity and velocity are computed at the grid points (idx, jdy) and are designated by $\omega_{i,j}$ $u_{i,j}$ and $v_{i,j}$ with $0 \leq i \leq m$ and $0 \leq j \leq n$. To approximate (2.1) we use a one step explicit method (Euler's method) to advance the solution in time, and approximate the advection term $\vec{u} \cdot \nabla\omega$ using a second order upwind differencing scheme due to Colella [5]. (We believe the results are relatively independent of the advection scheme used.) The second derivative term was approximated by central differences. Our scheme is thus expressed by

$$\frac{\omega_{i,j}^{k+1} - \omega_{i,j}^{k}}{dt} = A_{i,j}(\vec{u}^{k}, \omega^{k}) + D_y^{+} D_y^{-} \omega_{i,j}^{k}$$

where $A_{i,j}(\vec{u}^{k}, \omega^{k})$ is the second order approximation to the convective term in (2.1) and

$$D_y^{+} D_y^{-} \omega_{i,j}^{k} = \frac{\omega_{i,j+1}^{k} - 2\omega_{i,j}^{k} + \omega_{i,j-1}^{k}}{dy^2} \qquad (2.9)$$

To construct u we use the trapezoidal rule to approximate the integral in (2.3). We assume no vorticity above the line $y = y_{\infty}$, and hence $u = U_0$ for points above this line. For the remaining points ($j \leq n$) we use

$$u_{i,j} = U_0 + \sum_{p=j}^{n} \frac{(\omega_{i,p} + \omega_{i,p+1})}{2} dy \qquad (2.10)$$

v is approximated using

$$v_{i,j} = -\sum_{p=1}^{j} \frac{1}{2}\left(\frac{u_{i+1,p} - u_{i-1,p}}{2dx} + \frac{u_{i+1,p+1} - u_{i-1,p+1}}{2dx}\right) dy \qquad (2.11)$$

i.e. a trapezoidal rule approximation to (2.4) in which central differences are used to approximate the derivatives of u. At points on the left and right computational boundaries second order one sided differences are used in (2.11) instead of central differences.

It remains to specify the boundary conditions used for the vorticity. At the left edge of the computational boundary, the inflow side, we assumed that there is no vorticity immediately upstream. At the right edge of the computational boundary we assumed outflow boundary conditions - i.e. the vorticity is unspecified at points just outside of the computational domain. At the top and bottom of the domain we specified the normal derivative of ω. This data was incorporated into the difference stencil using the method of fictitious points. For points on such boundaries, the central difference occurring in (2.9) uses a point just outside the computational domain. This point is eliminated by using the normal derivative boundary condition. For example, at points on the plate, the second derivative term is approximated by

$$D_y^+ D_y^- \omega_{i,0} = \frac{2\omega_{i,1} - 2\omega_{i,0}}{dy^2} - \frac{2}{dy}\frac{\partial\omega}{\partial y}\Big|_{i,0}$$

For the data at the bottom we used

$$\frac{\partial\omega}{\partial y}\Big|_{i,0} = \frac{-1}{\nu}\sum_{p=0}^{m}\left[\frac{A_{i,p+1}(u^k, \omega^k) + A_{i,p}(u^k,\omega^k)}{2}\right]dy \tag{2.12}$$

i.e. a discrete analog of (2.8). At the top we used the boundary condition $\frac{\partial\omega}{\partial y}\Big|_{i,n} = 0$. This was chosen because it insures that the boundary condition $u = 0$ at $y = 0$ is satisfied exactly by the discrete scheme. This fact can be verified by directly calculating the difference in the velocity at the plate induced by ω^{k+1} and ω^k. The procedure is a discrete version of the arguments used to find the conditions that the time derivative of the constraint (2.7) vanish. An upper boundary condition appears, however, because the computational domain is finite. The dependence of the velocity at the plate upon the far field boundary condition illustrates the point that one must be careful about the numerical implementation of such schemes in unbounded domains. This problem with far field boundary conditions is also discussed within the context of the Navier Stokes equations in [1].

Our computational results correspond to a parameter selection of $\nu = .2$, $x_* = 2.0$, $y_* = 4.0$, $dx = dy = .1$, $\delta t = .0125$, and the onset velocity $U_0 = 1.0$. The values of parameters ν, U_0, x_*, and y_* were selected to insure that the majority of the vorticity was confined to the region $y \le y_*$ when $0 \le x \le x_*$. The timestep was chosen so that $\delta t < \min(\frac{dy^2}{2\nu}, \frac{dx}{U_0}, \frac{dy}{U_0})$. No instability was observed for this choice of parameters.

The use of the boundary condition (2.8) assumes that the initial vorticity induces a velocity field which satisfies all of the boundary conditions. The initial distribution of vorticity used,

$$\omega_{i,0}^{0} = \frac{U_0}{2dy} \qquad i = 0..m$$

$$\omega_{i,j}^{0} = 0 \qquad i = 1..m \qquad 0 < j \leq n$$

satisfies this assumption. This choice of initial conditions corresponds to impulsively starting the fluid at time t=0 with a velocity U_0.

The development of the boundary layer is depicted in figures 2.1(a) to 2.1(c). At time $t = 0$ the velocity is uniform with value U_0 except at the surface of the plate where it is zero. At a later time, $t = 2.5$, a typical boundary layer profile is emerging near the leading edge. Further down the plate the velocity field is still uniformly horizontal, as the upstream influence is not felt. The fluid is retarded, however, because of the diffusion of vorticity from the wall. At a much later time $t = 10.0$ the solution has reached steady state and the boundary layer profile is depicted in figure 2.1(c).

A steady state solution of the equations is the Blasius solution. This is a solution which is self similar with a similarity variable $\eta = y(\frac{U}{x\nu})^{\frac{1}{2}}$, i.e. the steady state solution (u,v) is given by

$$u(x,y) = u_b(\eta) \qquad v(x,y) = v_b(\eta)$$

where $(u_b(\eta),v_b(\eta))$ is the Blasius solution. A comparison of the steady state profile and the Blasius solution obtained from [10] is given in figure 2.2. In this figure, the u component of the computed velocity is shown using arrows, while the u component for the corresponding Blasius solution is denoted by the solid line. The solution is plotted for positions x=.5, x=1.0 and x=1.5 along the plate. The difference between the computed solution and the Blasius solution is extremely small.

To see more clearly the accuracy with which the Blasius profile is approximated, we show in figure 2.3 a plot of the difference between the computed u velocity and the u velocity of the Blasius solution. The results are plotted verses the similarity variable η so that each profile can be compared with the other. The velocity profile which is most in error is that near the leading edge of the plate, x=.5. This is to be expected since the number of points per profile is less at this station than at stations further down the plate. However, the maximum relative error of all the profiles is less than 1%.

These results demonstrate that it is possible to incorporate the boundary condition (2.8) into a numerical scheme and obtain accurate answers. Furthermore, this incorporation can be done within the framework of an explicit method. (There has been some doubt about the possibility of this.) The use of an explicit method is limiting because of the time step restriction. A semi-implicit finite difference scheme has been developed and used to obtain results which agree with those given above. We chose not to discuss the scheme because it is a bit more difficult to describe and not any more illuminating as an example than the purely explicit scheme.

We conclude this section by discussing the relation between a method introduced by Chorin for solving the Prandtl boundary layer equations and an analogous finite difference scheme. In [4], Chorin uses the method of fractional steps and vorticity boundary creation to solve equations (2.1)-(2.5). The discretization used is based on computational elements which are segments of vortex sheets. The basic timestep, ignoring the precise implementation details, are as follows: An approximate solution of the inviscid equations is advanced one time-step (equation 2.1 without ν). This leads to a vorticity distribution which induces a velocity field which does not satisfy the tangential boundary condition on the plate. If the velocity field at location idx on the plate is U_i, then a sheet of strength $2U_i$ is created on the boundary. This sheet is then allowed to participate, with the sheets already in the fluid, in a random walk. This random walk approximates one step of the evolution of the viscous terms of the equation.

Consider the following fractional step scheme which uses the finite difference approximations given previously,

$$\frac{\tilde{\omega}^{k+1}_{i,j} - \omega^{k}_{i,j}}{\delta t} = -\, A_{i,j}(\vec{u}^{k},\omega^{k}) \tag{2.13}$$

$$\frac{\omega^{k+1}_{i,j} - \tilde{\omega}^{k+1}_{i,j}}{\delta t} = \nu D^{+}_{y} D^{-}_{y} \tilde{\omega}^{k+1}_{i,j} \tag{2.14}$$

The normal derivative boundary condition

$$\frac{\partial \tilde{\omega}}{\partial y}\Big|_{i,0} = \frac{-1}{\nu} \sum_{p=0}^{n} [\frac{A_{i,p+1}(\vec{u}^{k}, \omega^{k}) + A_{i,p}(\vec{u}^{k},\omega^{k})}{2}] dy \tag{2.15}$$

is incorporated into the finite difference stencil occurring in (2.14). The upper boundary condition is $\frac{\partial \tilde{\omega}}{\partial y}|_{i,n} = 0$. One can show, as with the explicit scheme given previously, that if the velocity field induced by ω^{k} satisfies (2.5) then the velocity induced by ω^{k+1} satisfies (2.5) as well.

The effect of the boundary condition (2.15) is to induce vorticity on the boundary of the computational domain. One can isolate the amount of vorticity induced by this boundary condition by considering one step of equation (2.14) using zero initial data and boundary condition (2.15). This is the vorticity $\bar{\omega}$ defined by

$$\frac{\bar{\omega}_{i,j} - \omega_{0_{i,j}}}{\delta t} = \nu D^{+}_{y} D^{-}_{y} \omega_{0_{i,j}}$$

with initial condition $\omega_{0_{i,j}} = 0$ and boundary condition (2.15). Working through the algebra we find that $\bar{\omega}$ is non-zero only at the lower boundary, and is given by

$$\bar{\omega}_{i,0} = -\frac{2\delta t}{dy} \sum_{p=0}^{n} [\frac{A_{i,p+1}(\vec{u}^{k}, \omega^{k}) + A_{i,p}(\vec{u}^{k},\omega^{k})}{2}] dy \tag{2.16}$$

If we use the expression for $A_{i,j}$ from (2.13) in (2.16) we find that

$$\bar{\omega}_{i,0} = \frac{2}{dy}\sum_{p=0}^{n}[\frac{(\tilde{\omega}_{i,p+1}^{k+1} - \omega_{i,p+1}^{k})+(\tilde{\omega}_{i,p}^{k+1} - \omega_{i,p}^{k})}{2}]dy$$

$$= \frac{2}{dy}(U_0 + \sum_{p=0}^{n}[\frac{\tilde{\omega}_{i,p+1}^{k+1} + \tilde{\omega}_{i,p}^{k+1}}{2}]dy)$$

$$= \frac{2U_i}{dy} \tag{2.17}$$

Here we are making the assumption that $\omega_{i,j}^{k}$ satisfies (2.5) discretely, i.e.

$$U_0 + \sum_{j=0}^{n}\frac{(\omega_{i,p+1}^{k} + \omega_{i,p}^{k})}{2}dy = 0$$

The quantity U_i in (2.17) is the u velocity at the point idx on the plate which is induced by $\tilde{\omega}^{k+1}$. From the definition of $\tilde{\omega}^{k+1}$ we see that this is the slip induced by one step of Euler flow. The vorticity $\frac{2U_i}{dy}$ corresponds to a computational element which induces a jump in velocity of $2U_i$, i.e. a sheet with strength $2U_i$. This is precisely the strength of the vortex sheet which is introduced by Chorin in his scheme. Thus, there is exact agreement in the amount of vorticity which is added at each time step. We believe that this is a remarkable result in light of the difference in the types of discretizations used.

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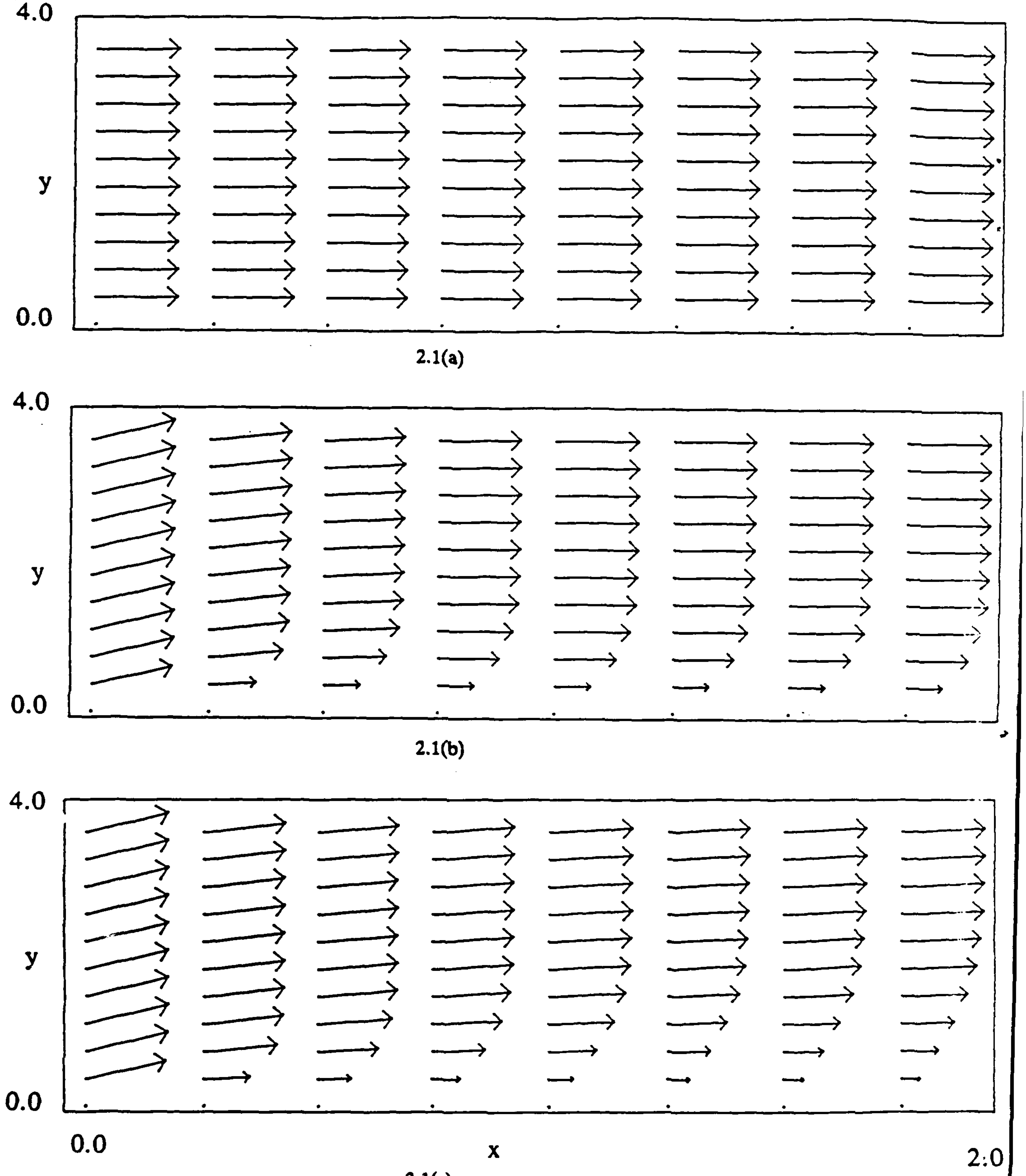

2.1(a)

2.1(b)

2.1(c)

Figure 2.1(a)-(c)
Velocity vector plots at times t = 0.0, t = 2.5 and t=10.0
in (a)-(c) respectively

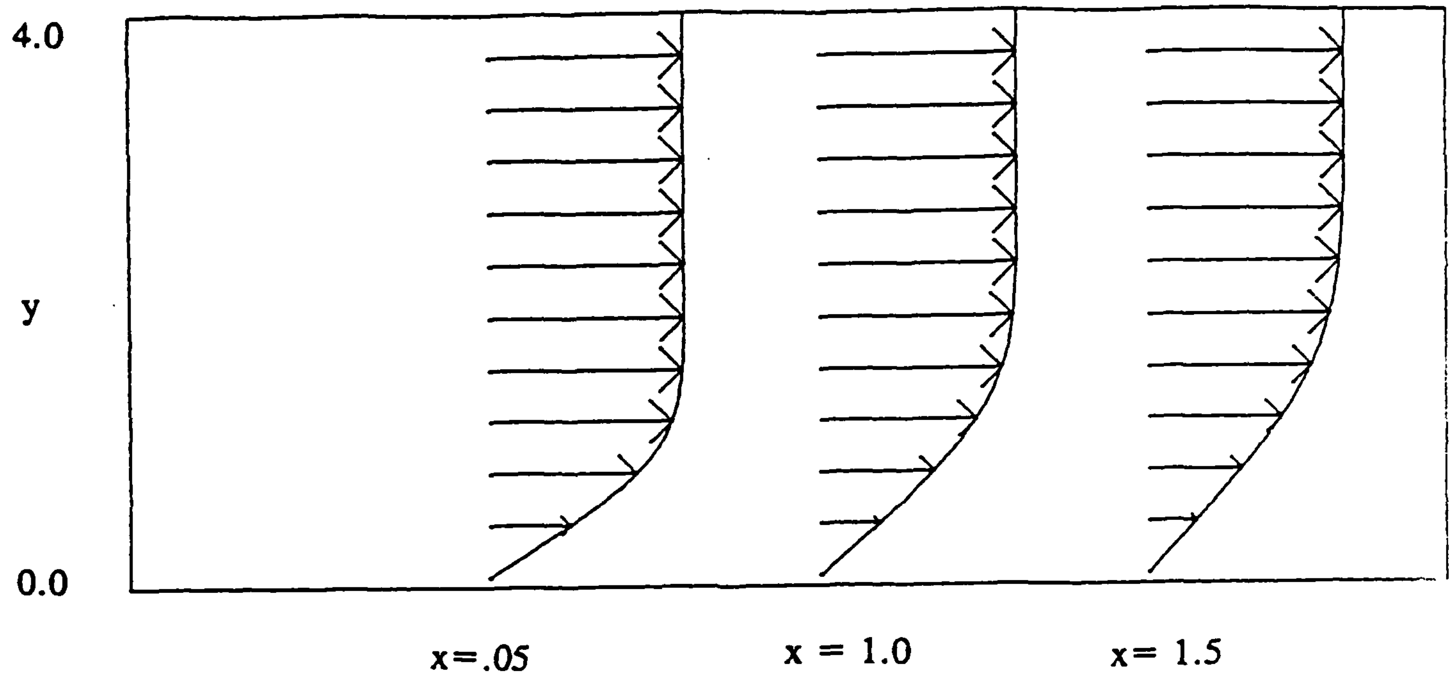

Figure 2.2
Comparison of the computed u velocity component (arrows) with the Blasius solution (solid line).

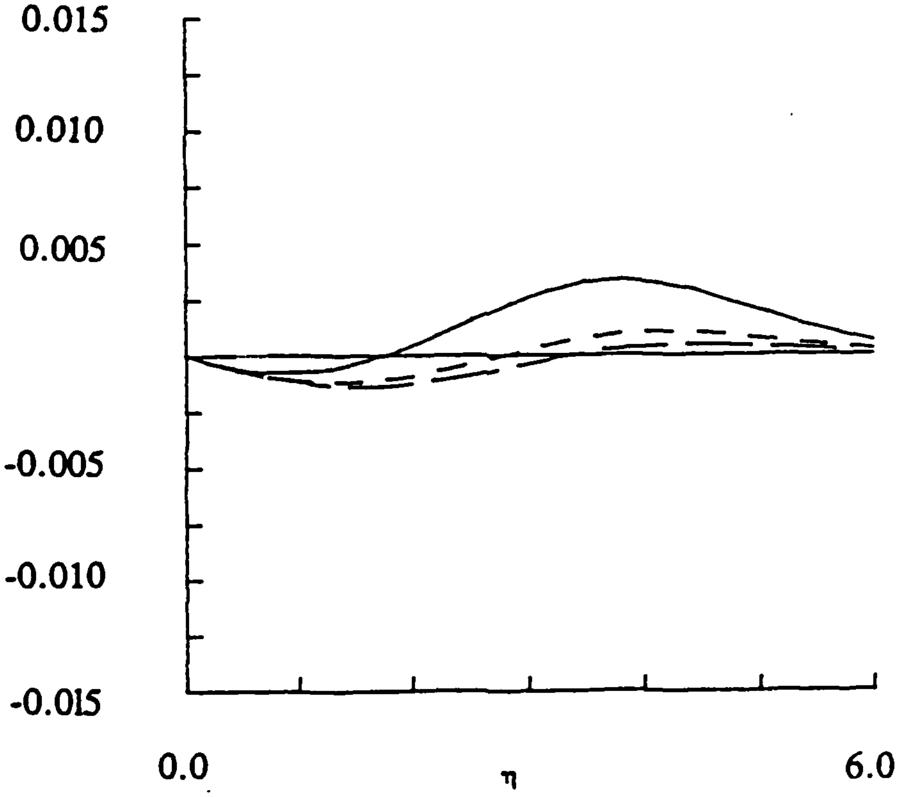

Figure 2.3
Difference between the computed u velocity component and the Blasius solution at stations x´= 0.5 (solid line), x = 1.0 (short dashed line), x = 1.5 (long dashed line).

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